I. Experiments, outcomes, sample spaces, events

Terminology

- **Experiment**: A process leading to exactly one of various possible outcomes.
- **Outcome (a.k.a. simple event, or sample point)**: The things that can happen in an experiment.
- **Sample space (a.k.a. probability space)**: The set of all possible outcomes. Sometimes denoted S.
- **Event**: A subset of the sample space, that is, a set of some of the outcomes. Sometimes denoted A.

Key formula

- The probability of event A is

$$P(A) = \frac{\#\text{ of outcomes in the event } A}{\#\text{ of outcomes in the sample space } S}$$

Example I

Roll two dice and see if the sum showing on the two dice is 10. Identify the experiment, all the outcomes, and the event we're interested in. Find the probability of the event.

- **Experiment**: Rolling the two dice.
- **Outcomes**: Think of one die as red and one die as blue. Then there are $\boxed{36}$ outcomes: $\boxed{S = \{1-1, 1-2, \ldots, 6-6\}}$.
- **Event**: They add to 10: $\boxed{A = \{4-6, 5-5, 6-4\}}$

$$P(A) = \frac{\#\text{ of outcomes in the event } A}{\#\text{ of outcomes in the sample space } S} = \frac{3}{36} = \boxed{\frac{1}{12}}$$

Example II

Flip a coin three times and see if we get exactly two heads. Identify the experiment, all the outcomes, and the event we're interested in. Find the probability of the event.

- **Experiment**: Flipping the coin.
- **Outcomes**: There are $\boxed{8}$ outcomes: $\boxed{S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}}$.
- **Event**: $\boxed{A = \{HHT, HTH, THH\}}$

$$P(A) = \frac{\#\text{ of outcomes in the event } A}{\#\text{ of outcomes in the sample space } S} = \boxed{\frac{3}{8}}$$

Example III

Draw cards repeatedly, without replacement, from a standard 52-card deck until we find the ace of spades. What is the probability that we draw between 20 and 30 cards, inclusive? Identify the experiment, all the outcomes, and the event we're interested in.

- **Experiment**: Drawing the cards.
- **Outcomes**: We could find the ace of spaces as anywhere from the 1st through the 52nd card, so there are $\boxed{52}$ outcomes.
- **Event**: $\boxed{A = \{20, 21, \cdots, 30\}}$

$$P(A) = \frac{\text{\# of outcomes in the event } A}{\text{\# of outcomes in the sample space } S} = \boxed{\frac{11}{52}}$$

Example IV

Roll two dice and see if the sum is a prime number. Identify the experiment, all the outcomes, and the event we're interested in. Find the probability of the event.

- **Experiment**: Rolling the two dice.
- **Outcomes**: Think of one die as red and one die as blue. Then there are $\boxed{36}$ outcomes: $\boxed{S = \{1-1, 1-2, \ldots, 6-6\}}$.
- **Event**: They add to 2, 3, 5, 7, or 11:

$$\begin{array}{rcl} 2 & : & 1-1 \\ 3 & : & 1-2, 2-1 \\ 5 & : & 1-4, 2-3, 3-2, 4-1 \\ 7 & : & 1-6, 2-5, 3-4, 4-3, 5-2, 6-1 \\ 11 & : & 5-6, 6-5 \end{array}$$

$$P(A) = \frac{\#\text{ of outcomes in the event } A}{\#\text{ of outcomes in the sample space } S} = \frac{15}{36} = \boxed{\frac{5}{12}}$$

Example V

Flip a coin repeatedly until we get a head. What is the probability that we flip at least three times? Identify the experiment, all the outcomes, and the event we're interested in. Find the probability of the event.

- **Experiment**: Flipping the coin.
- **Outcomes**: There are infinitely many outcomes: $\boxed{S = \{H, TH, TTH, TTTH, \dots\}}$.
- **Event**: $\boxed{A = \{TTH, TTTH, TTTTH, \dots\}}$

We can't do this by counting.

$$\begin{aligned} P(A) &= P(TTH) + P(TTTH) + P(TTTTH) + \cdots \\ &= \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\ &= \frac{\text{first term}}{1 - \text{ common ratio}} \\ &= \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \boxed{\frac{1}{4}} \end{aligned}$$

Alternate solution:

$$\begin{aligned} P(A) &= 1 - P\left(A^C\right) \\ &= 1 - [P(H) + P(TH)] \\ &= 1 - \left[\frac{1}{2} + \frac{1}{4}\right] = \boxed{\frac{1}{4}} \end{aligned}$$

Example VI

Draw a five-card poker hand from a standard 52-card deck. What is the probability that we draw all spades? Identify the experiment, all the outcomes, and the event we're interested in.

- **Experiment**: Drawing the cards.
- **Outcomes**: All the possible hands:

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{50!}{47! \cdot 5!} = \boxed{\binom{52}{5}}$$

- **Event**: All ways to get 5 spades:

$$\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{13!}{8! \cdot 5!} = \boxed{\binom{13}{5}}$$

$$P(A) = \frac{\#\text{ of outcomes in the event } A}{\#\text{ of outcomes in the sample space } S} = \boxed{\frac{\binom{13}{5}}{\binom{52}{5}}}$$