IX. Inhomogeneous equations: undetermined coefficients

Lesson Overview

- To solve the (linear, second-order, inhomogeneous, constant coefficient) differential equation

$$ay'' + by' + cy = g(t)$$

  first solve the homogeneous equation

$$ay'' + by' + cy = 0$$

  by the methods of the previous lecture.

- Then find a particular solution to the inhomogenous equation

$$ay'' + by' + cy = g(t)$$

  using undetermined coefficients. This means you guess something that looks like $g(t)$, but has generic coefficients. Then you plug it in and solve for the coefficients.

$g(t)$	Guess for y_{par}
ke^{rt}	Ae^{rt}
polynomial	$At^2 + Bt + C$ (same degree as $g(t)$)
$k \sin 5t$	$A \sin 5t + B \cos 5t$
Combinations above.	Combinations above.

- If any term of your guess for y_{par} looks like any term of y_{hom}, then multiply your whole guess by t.

- Solve for constants after finding y_{par}.

- If $g(t) = \ln t, \tan t$, etc., then abandon undetermined coefficients. Use variation of parameters (next lecture).

Example I

Find the general solution to the differential equation:

$$y'' - 3y' + 2y = 4e^{3t}$$

$r = 1, 2 \Longrightarrow y_{\text{hom}} = c_1 e^t + c_2 e^{2t}$

Guess: $y_{\text{par}} =$ (something that "looks like" $g(t)$) $= Ae^{3t}$. (A to be determined.) Plug in:

$$\begin{aligned} y'_{\text{par}} &= 3Ae^{3t} \\ y''_{\text{par}} &= 9Ae^{3t} \\ 9Ae^{3t} - 3\left(3Ae^{3t}\right) + 2\left(Ae^{3t}\right) &= 4Ae^{3t} \\ A &= 2 \end{aligned}$$

General Solution: $\boxed{y_{\text{gen}} = c_1 e^t + c_2 e^{2t} + 2e^{3t}}$.

Use IC to solve for constants after finding y_{par}.

Example II

Find the general solution to the differential equation:

$$y'' - 3y' + 2y = 4t^2$$

$r = 1, 2 \Longrightarrow y_{\text{hom}} = c_1 e^t + c_2 e^{2t}$

Guess: $y_{\text{par}} =$ (something that "looks like" $g(t)$)

$= At^2 + Bt + C$. Plug in:

$$\begin{aligned} y_{\text{par}} &= At^2 + Bt + C \\ y'_{\text{par}} &= 2At + B \\ y''_{\text{par}} &= 2A \\ y'' - 3y' + 2y &= 2A - 6At - 3B + 2At^2 + 2Bt + 2C = 4t^2 \end{aligned}$$

$$\begin{array}{lll} \underline{t^2}: & 2A = 4 & \Longrightarrow A = 2 \\ \underline{t}: & 2B - 6A = 0 & \Longrightarrow B = 6 \\ \underline{\text{const}}: & 2A - 3B + 2C = 0 & \Longrightarrow C = 7 \end{array}$$

$$\begin{aligned} y_{\text{par}} &= 2t^2 + 6t + 7 \\ y_{\text{gen}} &= \boxed{c_1 e^t + c_2 e^{2t} + 2t^2 + 6t + 7} \end{aligned}$$

Example III

Find the general solution to the differential equation:

$$y'' - 3y' + 2y = 5\cos 2t$$

$r = 1, 2 \Longrightarrow y_{\text{hom}} = c_1 e^t + c_2 e^{2t}$

Guess: $y_{\text{par}} =$ (something that "looks like" $g(t)$)

$= A\cos 2t + B\sin 2t$. Plug in:

$$\begin{aligned}
y_{\text{par}} &= A\cos 2t + B\sin 2t \\
y'_{\text{par}} &= -2A\sin 2t + 2B\cos 2t \\
y''_{\text{par}} &= -4A\cos 2t - 4B\sin 2t \\
y'' - 3y' + 2y &= -4A\cos 2t - 4B\sin 2t - 3\left(-2A\sin 2t + 2B\cos 2t\right) + 2\left(A\cos 2t + B\sin 2t\right) \\
&= (-2A - 6B)\cos 2t + (6A - 2B)\sin 2t = 5\cos 2t \\
-2A - 6B &= 5 \\
6A - 2B &= 0 \Longrightarrow B = 3A \\
-2A - 6(3A) &= 5 \Longrightarrow -20A = 5 \Longrightarrow A = -\frac{1}{4} \Longrightarrow B = -\frac{3}{4} \\
y_{\text{par}} &= -\frac{1}{4}\cos 2t - \frac{3}{4}\sin 2t \\
y_{\text{gen}} &= \boxed{c_1 e^t + c_2 e^{2t} + -\frac{1}{4}\cos 2t - \frac{3}{4}\sin 2t}
\end{aligned}$$

Example IV

Find the general solution to the differential equation:

$$y'' - 3y' + 2y = 5e^t$$

$r = 1, 2 \Longrightarrow y_{\text{hom}} = c_1 e^t + c_2 e^{2t}$

Guess: $y_{\text{par}} = Ae^t$. This is doomed to fail, because this y_{par} is a copy of y_{hom}!

Use $y_{\text{par}} = Ate^t$ instead.

$$\begin{aligned}
y'_{\text{par}} &= Ate^t + Ae^t \\
y''_{\text{par}} &= Ate^t + Ae^t + Ae^t \\
&= Ate^t + 2Ae^t \\
Ate^t + 2Ae^t - 3\left(Ate^t + Ae^t\right) + 2Ate^t &= 5e^t \quad \{t \text{ terms cancel.} \quad \} \\
-Ae^t &= 5e^t \\
A &= -5 \\
y_{\text{par}} &= -5te^t \\
y_{\text{gen}} &= \boxed{c_1 e^t + c_2 e^{2t} - 5te^t}
\end{aligned}$$

Example V

Give an appropriate form for the particular solution to the differential equation:

$$y'' - 3y' + 2y = t^4 + 7e^{3t}$$

$r = 1, 2 \Longrightarrow y_{\text{hom}} = c_1 e^t + c_2 e^{2t}$

Strategy: Solve $L[y_{p_1}] = t^4$ by guessing $y_{p_1} := At^4 + Bt^3 + Ct^2 + Dt + E$. Then solve $L[y_{p_2}] = 7e^{3t}$ by guessing $y_{p_1} := Fe^{3t}$. Then let $y_p := y_{p_1} + y_{p_2}$, so $L[y_p] = t^4 + e^{3t}$.