XXVII. Partial differential equations: The heat equation

Lesson Overview

- A partial differential equation is an equation that relates a function $u(x,t)$ and one or more of its partial derivatives.
- The most common ones relate u_{xx} with either u_t or u_{tt}.
- A boundary value problem is a partial differential equation packaged with initial conditions and boundary conditions (which are often zero):

$$\begin{aligned} u(x,0) &= f(x) \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

Common partial differential equations

- Heat Equation: $u_t = \alpha^2 u_{xx}$.

  $u(x,t)$ gives the temperature of a rod at position x at time t.
- Wave Equation: $u_{tt} = c^2 u_{xx}$.

  $u(x,t)$ gives the displacement of a string from 0 at position x at time t.
- Laplace's Equation: $u_{xx} + u_{tt} = 0$.

  This is used in complex analysis.

Example I

Give a physical interpretation for each element of the boundary value problem associated with the heat equation:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(x,0) &= f(x), 0 \leq x \leq L \\ u(0,t) &= 0, t \geq 0 \\ u(L,t) &= 0, t \geq 0 \end{aligned}$$

- α^2 is a constant of thermal diffusivity that depends on the type of material in the rod. (Typical values are found in the chart on page 612 in Boyce. Silver, which is highly conductive, has $\alpha^2 = 1.71$. Cast iron has $\alpha^2 = 0.12$. Water has $\alpha^2 = 0.00144$.)
- The experimenter must also measure and tell us the <u>initial temperature</u> everywhere in the rod:

  $$u(x,0) = f(x), 0 \leq x \leq L$$

- The experimenter must tell us what will happen at the ends of the rod. A common plan is to pack the ends in ice so that they stay at temperature 0. (So heat will flow out of the rod through the ends and we expect that it will gradually cool down.)

Example II

Give a physical interpretation for each element of the boundary value problem associated with the heat equation:

$$\begin{aligned} u_t &= \frac{1}{4}u_{xx} \\ u_x(0,t) &= 0, t \geq 0 \\ u_x(1,t) &= 0, t \geq 0 \\ u(x,0) &= 1 - x^2, 0 \leq x \leq 1 \end{aligned}$$

- α^2 is a constant of thermal diffusivity that depends on the type of material in the rod.
- The experimenter must also measure and tell us the initial temperature everywhere in the rod:

$$u(x,0) = f(x), 0 \leq x \leq L$$

- Before, the ends of the rod were packed in ice, so they stayed at temperature $u = 0$ and heat left the rod through the ends. Now, the ends of the rod are insulated, so that no heat can leave the rod. No heat leaving means that the partial derivative is $u_x = 0$.)

Example III

Give a physical interpretation for each element of the boundary value problem associated with the wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x), 0 \leq x \leq L \\ u(0,t) &= 0, t \geq 0 \\ u(L,t) &= 0, t \geq 0 \end{aligned}$$

- c^2 is a constant of elasticity that depends on the type of material in the string.

- The experimenter must also measure and tell us the initial position of the string everywhere:

$$u(x,0) = f(x), 0 \le x \le L$$

- The ends of the string are pinned down.

Example IV

Check that each of the following functions is a solution to the wave equation $u_{tt} = 9u_{xx}$:

A. $u(x,t) = \cos(x)\sin(3t)$

B. $u(x,t) = \cos(x)\cos(3t)$

C. $u(x,t) = e^t e^{\frac{x}{3}}$

D. $u(x,t) = e^{3t} e^x$

E. $u(x,t) = \sin(3t + x)$

Example IV

$u_{tt} = 9u_{xx}$

A. $u(x,t) = \cos(x)\sin(3t)$

$$\begin{aligned} u_x &= -\sin(x)\sin(3t) \\ u_{xx} &= -\cos(x)\sin(3t) \\ u_t &= 3\cos(x)\cos(3t) \\ u_{tt} &= -9\cos(x)\cos(3t) \\ u_{tt} &= 9u_{xx}\checkmark \end{aligned}$$

B. Similar.

C. Similar.

D. Similar.

E. $u_{tt} = -9\sin(3x+t) = 9u_{xx}$

Example V

Check that each of the following functions is a solution to the heat equation $u_t = 4u_{xx}$. Determine which one also satisfies the boundary condition $u(0,t) = 0$:

A. $u(x,t) = e^x e^{4t}$

B. $u(x,t) = e^{\frac{x}{2}} e^t$

C. $u(x,t) = (\sin x)e^{-4t}$

D. $u(x,t) = \left(\cos\frac{x}{2}\right) e^{-t}$

Example V

$u_t = 4u_{xx}, u(0,t) = 0$

Check each one and then plug in $x = 0$ to each one. Only $\boxed{u(x,t) = (\sin x)e^{-4t}}$ satisfies $u(0,t) = 0$.