XXV. Numerical techniques: Runge-Kutta/improved Euler method

Lesson Overview

- Runge-Kutta, also known as the improved Euler method, is a way to find numerical approximations for initial value problems that we can't solve analytically.
- It is more sophisticated than Euler's method.
- It is the fundamental algorithm used in most professional software to solve differential equations.
- We will learn the order 2 Runge-Kutta algorithm.

Runge-Kutta order 2 algorithm

- Start with an initial value problem in the form $y'(t) = f(t, y), y(t_0) = y_0$.
- Choose a step size h (usually given).
- Start at (t_0, y_0) and make iterative steps:

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + h\frac{k_1 + k_2}{2} \\ \text{where} \quad k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + h, y_n + hk_1) \end{aligned}$$

- Continue until you arrive at the value of t for which you need to approximate $y(t)$.

Example I

Use Runge-Kutta with step size $h = 0.1$ to estimate $y(0.1)$ in the initial value problem $y' = 1 + t - y, y(0) = 1$.

$$
\begin{aligned}
y(t_{n+1}) &= y_n + h\frac{k_1 + k_2}{2} \\
\text{where} \quad k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + h, y_n + hk_1) \\
(t_0, y_0) &= (0, 1) \\
k_1 &= f(0, 1) = 0 \\
k_2 &= f(0.1, 1 + 0.1(0)) \\
&= f(0.1, 1) \\
&= 0.1 \\
y(0.1) &= 1 + 0.1\left(\frac{0 + 0.1}{2}\right) \\
&= 1.005 \\
(t_1, y_1) &= (0.1, \boxed{1.005})
\end{aligned}
$$

Example II

Use Runge-Kutta with step size $h = 0.1$ to estimate $y(0.2)$ in the initial value problem $y' = 1 + t - y, y(0) = 1$.

$$\begin{aligned}
y(t_{n+1}) &= y(t_n) + h\frac{k_1+k_2}{2} \\
\text{where} \quad k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + h, y_n + hk_1) \\
(t_1, y_1) &= (0.1, 1.005) \\
k_1 &= f(0.1, 1.005) = 1 + 0.1 - 1.005 \\
&= 0.095 \\
k_2 &= f(0.2, 1.005 + 0.1(0.095)) \\
&= f(0.2, 1.005 + .0095) \\
&= f(0.2, 1.0145) \\
&= 1 + 0.2 - 1.0145 \\
&= 0.1855 \\
y(0.2) &= 1.005 + 0.1\left(\frac{0.095 + 0.1855}{2}\right) \\
&= 1.005 + 0.1\left(\frac{0.2805}{2}\right) \\
&= 1.005 + 0.1\,(0.14025) \\
&= 1.005 + 0.014025 \\
&= \boxed{1.019025}
\end{aligned}$$

Example III

Solve the initial value problem

$$y' = 1 + t - y, y(0) = 1$$

analytically. Compute $y(0.2)$ and compare the answer with the result given by Runge-Kutta above.

$$\begin{aligned}
y' + y &= 1 + t \quad \{I(t) = e^t \qquad \} \\
y'e^t + ye^t &= e^t + te^t \\
(ye^t)' &= e^t + te^t \\
ye^t &= te^t + C \\
y &= t + Ce^{-t} \\
1 &= 0 + C \\
C &= 1 \\
y &= \boxed{t + e^{-t}} \\
y(0.2) &= 0.2 + e^{-0.2} \\
&\approx \boxed{1.01873} \\
\text{Runge-Kutta:} \quad y(0.2) &\approx \boxed{1.019025} \\
\text{Euler:} \quad y(0.2) &\approx \boxed{1.01}
\end{aligned}$$

We were off by about $1.019 - 1.0187 = \boxed{0.0003}$. RK is more work, but it is much more accurate than Euler!

Example IV

Use Runge-Kutta with step size $h = 0.1$ to estimate $y(0.1)$ in the initial value problem $y' = t^2 + y^2, y(0) = 1$.

$$\begin{aligned}
y(t_{n+1}) &= y(t_n) + h\frac{k_1 + k_2}{2} \\
\text{where} \quad k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + h, y_n + hk_1) \\
(0, 1) &\to k_1 = 1, k_2 = f(0.1, 1.1) = 0.01 + 1.21 = 1.22 \\
y(0.1) &= 1 + 0.1\frac{1 + 1.22}{2} = \boxed{1.111}
\end{aligned}$$

Example V

Use Runge-Kutta with step size $h = 0.1$ to estimate $y(0.2)$ in the initial value problem $y' = t^2 + y^2, y(0) = 1$.

$$
\begin{aligned}
y(t_{n+1}) &= y(t_n) + h\frac{k_1 + k_2}{2} \\
\text{where} \quad k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + h, y_n + hk_1) \\
y(0.1) &= 1.111 \\
(0.1, 1.111) &\rightarrow k_1 = 0.01 + 1.23432 = 1.24432 \\
& \quad\; k_2 = f(0.2, 1.111 + 0.1(1.24432)) = 0.04 + 1.52629 = 1.56629 \\
y(0.2) &= 1.111 + 0.1\frac{1.24432 + 1.56629}{2} \\
&= 1.111 + 0.140531 \\
&= \boxed{1.25153}
\end{aligned}
$$