XIV. Series solutions: regular singular points

Lesson Overview

- We want to find series solutions $\sum_{n=1}^{\infty} a_n x^n$ to the differential equation:

$$\begin{aligned} P(x)y'' + Q(x)y' + R(x)y &= 0 \\ y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y &= 0 \end{aligned}$$

- But if $P(0) = 0$, then $x_0 = 0$ is called a <u>singular point</u> and the strategy doesn't work.

Definition: Pole of order n

- We say $f(x)$ has a <u>pole of order n</u> at $x_0 = 0$ if f has a series whose first term is $\frac{1}{x^n}$.
- We say $x_0 = 0$ is a <u>regular</u> singular point if
  1. $\frac{Q}{P}$ has a pole of order <u>at most 1</u> at 0, and
  2. $\frac{R}{P}$ has a pole of order <u>at most 2</u> at 0.
- If $x_0 = 0$ is a <u>regular</u> singular point, then we can use a solution of the form:

$$y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) = x^r \sum_{n=0}^{\infty} a_n x^n$$

- $a_0 \neq 0$

Solving around regular singular points

- Plug the series into the differential equation.
- You'll get an indicial equation for r, which will have two roots.
- If the difference between the roots is an integer, then you can find a solution for the larger root only.
- If the difference between the roots is not an integer, then you can find a solution for each of the two roots.

Example I

Find the order of the pole at 0 for each of the following functions:

$$\frac{e^x}{x};\quad \frac{\sin x}{x};\quad \frac{1}{x^2};\quad 5x + \frac{4}{x^3} - \frac{2}{x}$$

- $\frac{e^x}{x}$ has a pole of order $\boxed{1}$ at 0.
- $\frac{\sin x}{x}$ has a pole of order $\boxed{0}$ at 0.
- $\frac{1}{x^2}$ has a pole of order $\boxed{2}$ at 0.
- $f(x) = 5x + \frac{4}{x^3} - \frac{2}{x}$. f has a pole of order $\boxed{3}$ at 0.

Example II

Determine whether $x_0 = 0$ is a regular singular point for each of the following equations:

- $x^2y'' + (\sin x)y' + 3y = 0$
- $x^2y'' + (\cos x)y' + e^x y = 0$

- $x^2y'' + (\sin x)y' + 3y = 0$ has a $\boxed{\text{regular}}$ singular point at $x_0 = 0$.
- $x^2y'' + (\cos x)y' + e^x y = 0$ has an $\boxed{\text{irregular}}$ singular point at $x_0 = 0$, because $\frac{Q}{P}$ has a pole of order 2 there.

Example III

Find and solve the indicial equation for the differential equation:

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0$$

Note that $x_0 = 0$ is a regular singular point.

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0 \text{ (\underline{not} 1!)}}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=\boxed{2}}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$-y = \sum_{n=0}^{\infty} (-a_n x^{n+r})$$

$$2x^2y = \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = \sum_{n=\boxed{2}}^{\infty} 2a_{n-2} x^{n+r}$$

$$3xy' = \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r}$$

$$2x^2y'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r}$$

Example III

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0$$

$$\underbrace{[-1 + 3r + 2r(r-1)]a_0x^r}_{n=0} + \underbrace{[-1 + 3(r+1) + 2(r+1)r]a_1x^{r+1}}_{n=1} + \cdots$$

$$\cdots + \sum_{n=\boxed{2}}^{\infty} \left\{[-1 + 3(n+r) + 2(n+r)(n+r-1)]\, a_n + 2a_{n-2}\right\} x^{n+r} = 0$$

$a_0 \neq 0 \implies [-1 + 3r + 2r(r-1)] = 0$ {This is the <u>indicial equation</u>.}

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0$$

$$r = \frac{1}{2}, -1$$

Example IV

Find a solution to the differential equation above corresponding to $r = -1$.

$$\underbrace{[-1 + 3r + 2r(r-1)]a_0x^r}_{n=0} + \underbrace{[-1 + 3(r+1) + 2(r+1)r]a_1x^{r+1}}_{n=1} + \cdots$$

$$\cdots + \sum_{n=\boxed{2}}^{\infty} \left\{[-1 + 3(n+r) + 2(n+r)(n+r-1)]\, a_n + 2a_{n-2}\right\} x^{n+r} = 0$$

<u>$r = -1$</u>:

$$\begin{aligned}
\underline{x^r}: \quad [-1 + 3(-1) + 2(-1)(-2)]a_0 &= 0 \implies a_0 = \text{arbitrary} \\
\underline{x^{r+1}}: \quad [-1 + 3(0) + 2(0)(-1)]a_1 &= 0 \implies a_1 = 0 \\
[-1 + 3(n+r) + 2(n+r)(n+r-1)]\, a_n + 2a_{n-2} &= 0 \text{ for } n \geq \boxed{2} \\
[-1 + 3n - 3 + 2(n-1)(n-2)]\, a_n &= -2a_{n-2} \\
(2n^2 - 3n)a_n &= -2a_{n-2} \\
a_n &= \frac{-2a_{n-2}}{n(2n-3)} \text{ for } n \geq 2
\end{aligned}$$

Example IV

$$a_n = \frac{-2a_{n-2}}{n(2n-3)} \text{ for } n \geq 2$$

$$\begin{aligned}
n = 2: \quad a_2 &= \frac{-2a_0}{2 \cdot 1} \\
n = 3: \quad a_3 &= \leftrightsquigarrow a_1 = 0 \\
n = 4: \quad a_4 &= \frac{2^2 a_0}{2 \cdot 4 \cdot 1 \cdot 5} \\
a_6 &= \frac{-2^3 a_0}{2 \cdot 4 \cdot 6 \cdot 1 \cdot 5 \cdot 9} = \frac{-a_0}{3!\, 1 \cdot 5 \cdot 9} \\
a_8 &= \frac{a_0}{4!\, 1 \cdot 5 \cdot 9 \cdot 13}
\end{aligned}$$

$$\begin{aligned}
y_1 &= x^r \sum_{n=0}^{\infty} a_n x^n \\
&= x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \\
&= x^{-1} \left(a_0 - a_0 x^2 + \frac{a_0}{2!\, 1 \cdot 5} x^4 - \frac{a_0}{3!\, 1 \cdot 5 \cdot 9} x^6 + \frac{a_0}{4!\, 1 \cdot 5 \cdot 9 \cdot 13} x^8 + \cdots \right) \\
&= a_0 x^{-1} \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n! 1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)} \right) \quad \left\{ \text{(These are not the same } n\text{'s as above!)} \right\} \\
&= \boxed{x^{-1} + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{n! 1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}}
\end{aligned}$$

Note that $r = \frac{1}{2}$ would lead to a different solution.

Example V

Find and solve the indicial equation for the differential equation below. Determine which root(s) would lead to a valid solution.

$$x^2 y'' - 3xy' + (x+3)y = 0$$

Check: $\frac{Q}{P}$ has a pole of order 1, and $\frac{R}{P}$ has a pole of order 2, so $x_0 = 0$ is a regular singular point.

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \qquad \{\text{Assume } a_0 \neq 0. \}$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad 3y = \boxed{\sum_{n=0}^{\infty} 3a_n x^{n+r}}$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+r+1} = \boxed{\sum_{n=1}^{\infty} a_{n-1} x^{n+r}}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \qquad -3xy' = \boxed{\sum_{n=0}^{\infty} \left[-3(n+r) a_n x^{n+r}\right]}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \qquad x^2 y'' = \boxed{\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}}$$

$\underline{n=0}$: $[3-3r+r(r-1)]a_0 x^r = 0 \Longrightarrow \boxed{r^2 - 4r + 3 = 0} \Longrightarrow \boxed{r = 1, 3}$

Since these differ by an integer, only the larger one, $\boxed{r = 3}$ would lead to a valid solution.